Localized Excitations in a Dispersive Long Water-Wave System via an Extended Projective Approach

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By means of an extended projective approach, a new type of variable separation excitation with arbitrary functions of the (2+1)-dimensional dispersive long water-wave (DLW) system is derived. Based on the derived variable separation excitation, abundant localized coherent structures such as single-valued localized excitations, multiple-valued localized excitations and complex wave excitations are revealed by prescribing appropriate functions. – PACS numbers: 03.65.Ge, 05.45.Yv

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1. Introduction

In modern nonlinearity, the study of exact solutions and related issues of the construction of solutions to a wide class of nonlinear physical systems have become one of the most exciting and extremely active areas of current investigation [1 – 4]. Early in the study of soliton theory, the main interests of scientists were restricted to (1+1)-dimensional cases due to the difficulty of searching for higher-dimensional solitary wave solutions which are physically significant and localized in all directions. Recently, because of the rapid development of computer algebra and its mature application, the study of soliton theory concerning evolutional properties of localized coherent solition excitations in higher-dimensional soliton systems has attracted much more attention both for physicists and mathematicians. In this paper, we consider the localized excitation phenomena in the following celebrated (2+1)-dimensional dispersive long water-wave (DLW) system:

$$u_{ty} + v_{xx} + u_x u_y + u u_{xy} = 0, \qquad v_t + (uv)_x + u_{xxy} = 0. \tag{1}$$

The DLW system was first derived by Boiti et al. [5] as a compatibility set for a “weak” lax pair. In [6], Paquin and Winternitz showed that the symmetry algebra of (1) is infinite-dimensional with a Kac-Moody-Virasoro structure. Some special similarity solutions are also given in [6] by using symmetry algebra and classical theoretical analysis. The more general symmetry algebra, W_∞, is given in [7]. In [8], Lou gave nine types of two-dimensional similarity reductions and thirteen types of ordinary differential equation reductions. In [9], Lou has shown that (1) has no Painlevé property, though the system is Lax- or IST-integrable. Abundant propagating localized excitations were also derived by Tang and Lou [2] with the help of the Painlevé-Bäcklund transformation and a multilinear variable separation approach. However, to the best of our knowledge, its single-valued localized excitations, multiple-valued localized excitations and complex wave excitations via a projective approach were little reported in the preceding literature.

2. Exact Solutions to the (2+1)-Dimensional DLW System

In this section, we give some exact solutions to the DLW system, including solitary wave solutions, periodic wave solutions and Weierstrass function solutions.

Letting $f \equiv f(\xi(\chi))$, $g \equiv g(\xi(\chi))$, where $\xi \equiv \xi(\chi)$ is a undetermined function of the independent variables $\chi \equiv (x_0 = t, x_1, x_2, \cdots, x_m)$, the projective Riccati equation [10, 11] is defined by

$$f' = pfg, \quad g' = q + pg^2 - rf, \tag{2}$$

where $p^2 = 1$, q and r are two real constants. When $p = -1$ and $q = 1$, (2) reduces the coupled equations given in [10] and the following relation between f and g can be satisfied as $\delta = \pm 1$ and $q \neq 0$:

$$g^2 = -\frac{1}{p}\left[q - 2rf + \frac{r^2+\delta}{q}f^2\right]. \quad (3)$$

Equation (2) had been discussed in [11]. In this paper, we discuss other cases.

Lemma. If the condition of (3) holds with other choices of δ, the projective Riccati equation (2) has the following solutions:

(a) If $\delta = -r^2$, the Weierstrass elliptic function solution is admitted:

$$f = \frac{q}{6r} + \frac{2}{pr}\wp(\xi), \quad g = \frac{12\wp'(\xi)}{q + 12p\wp(\xi)}. \quad (4)$$

Here $p = \pm 1$, the Weierstrass elliptic function $\wp(\xi) = \wp(\xi; g_2, g_3)$ satisfies $\wp'^2(\xi) = 4\wp^3(\xi) - g_2\wp(\xi) - g_3$, and $g_2 = \frac{q^2}{12}$, $g_3 = \frac{pq^3}{216}$.

(b) If $\delta = -\frac{r^2}{25}$, the projective Riccati equation (2) has the Weierstrass elliptic function solution

$$f = \frac{5q}{6r} + \frac{5pq^2}{72r\wp(\xi)}, g = -\frac{q\wp'(\xi)}{\wp(\xi)(12\wp(\xi)+pq)}, \quad (5)$$

where $p = \pm 1$. Both q and r in (4) and (5) are arbitrary constants.

(c) If $\delta = h^2 - s^2$ and $pq < 0$, (2) has the solitary solution

$$f = \frac{q}{r + s\cosh(\sqrt{-pq}\xi) + h\sinh(\sqrt{-pq}\xi)}, \quad (6)$$

$$g = -\frac{\sqrt{-pq}}{p}\frac{s\sinh(\sqrt{-pq}\xi) + h\cosh(\sqrt{-pq}\xi)}{r + s\cosh(\sqrt{-pq}\xi) + h\sinh(\sqrt{-pq}\xi)},$$

where $p = \pm 1$, s and h are arbitrary constants.

(d) If $\delta = -h^2 - s^2$ and $pq > 0$, (2) possesses the trigonometric function solution

$$f = \frac{q}{r + s\cos(\sqrt{pq}\xi) + h\sin(\sqrt{pq}\xi)},$$
$$g = \frac{\sqrt{pq}}{p}\frac{\sin(\sqrt{pq}\xi) - h\cos(\sqrt{pq}\xi)}{r + s\cos(\sqrt{pq}\xi) + h\sin(\sqrt{pq}\xi)}, \quad (7)$$

where $p = \pm 1$, s and h are arbitrary constants.

(e) If $q = 0$, (2) has the rational solution

$$f = \frac{2}{pr\xi^2 + C_1\xi - C_2}, g = -\frac{2pr\xi + C_1}{(pr\xi^2 + C_1\xi - C_2)p}, \quad (8)$$

where C_1, C_2, and r are arbitrary constants, $p = \pm 1$.

We now introduce the mapping approach via the above projective Riccati equation. The basic idea of the algorithm is: Consider a nonlinear partial differential equation (NPDE) with independent variables $X \equiv (x_0 = t, x_1, x_2, \cdots, x_m)$ and the dependent variable $u \equiv u(X)$:

$$P(u, u_t, u_{x_i}, u_{x_i x_j} \cdots) = 0, \quad (9)$$

where P is a polynomial function of its argument and the subscripts denote the partial derivatives. We assume that its solution is written as the standard truncated Painlevé expansion, namely

$$u = A_0(X) + \sum_{i=1}^{n}[A_i(X)f(\xi(X)) + B_i(X)g(\xi(X))]f^{i-1}(\xi(X)). \quad (10)$$

Here $A_0(X)$, $A_i(X)$, $B_i(X)$ $(i = 1, \cdots, n)$ are arbitrary functions to be determined, and f, g satisfy the projective Riccati equation (2).

To determine u explicitly, one proceeds as follows: First, similar to the usual mapping approach, we can determine n by balancing the highest-order partial differential terms with the highest nonlinear terms in (9). Second, substituting (10) together with (2) and (3) into the given NPDE, collecting the coefficients of the polynomials of $f^i g^j$ and eliminating each of them, we can derive a set of partial differential equations for $A_0(X)$, $A_i(X)$, $B_i(X)$ $(i = 0, \cdots, n)$ and $\xi(X)$. Third, to calculate $A_0(X)$, $A_i(X)$, $B_i(X)$ $(i = 0, \cdots, n)$ and $\xi(X)$, we solve these partial differential equations. Finally, substituting $A_0(X)$, $A_i(X)$, $B_i(X)$ $(i = 0, \cdots, n)$, $\xi(X)$, and the solutions (4) – (8) into (10), one obtains solutions of the given NPDE.

Now, we apply the above mapping approach to the DLW system. According to the balancing procedure, (10) becomes

$$u = a_0 + a_1 f(\xi) + b_1 g(\xi), \quad (11)$$

$$v = A_0 + A_1 f(\xi) + A_2 f^2(\xi) + B_1 g(\xi) + B_2 f(\xi) g(\xi),$$

where a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2 and ξ are arbitrary functions of $\{x, y, t\}$ to be determined. Substituting (11) together with (2) and (3) into (1), collecting the coefficients of the polynomials of $f^i g^j$ $(i = 0, 1, 2, 3, 4,\ j = 0, 1)$ and setting each of the coefficients equal to zero, we can derive a set of partial differential equations for a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2 and ξ. It is difficult to obtain the general solutions of these algebraic equations based on the solutions of (2). Fortunately, in the special case if setting $\xi = \chi(x,t) + \varphi(y)$, where $\chi \equiv \chi(x,t)$, $\varphi \equiv \varphi(y)$ are two arbitrary variable separated functions of (x,t) and y, respectively, we can obtain solutions of (1).

Theorem. For the (2+1)-dimensional DLW system (1), there are five couples of variable separated solutions, related to the projective Riccati equation (2).

(a) For $\delta = -r^2$, the Weierstrass elliptic function solutions are

$$u_1 = \frac{\varepsilon\chi_{xx} - \chi_t}{\chi_x} + \varepsilon p\chi_x g(\xi), v_1 = -pr\chi_x\varphi_y f(\xi), \quad (12)$$

where $p = \pm 1$, $\varepsilon = \pm 1$, and f, g are expressed by (4).

(b) For $\delta = \frac{-r^2}{25}$, another set of Weierstrass elliptic function solutions is found:

$$\begin{aligned} u_2 &= \frac{\chi_{xx} - \chi_t}{\chi_x} + \varepsilon\frac{2r\sqrt{-6pq}}{5q}\chi_x f(\xi) + p\chi_x g(\xi), \\ v_2 &= -pr\chi_x\varphi_y f(\xi) + \frac{24pr^2}{25q}\chi_x\varphi_y f^2(\xi) \\ &\quad - \varepsilon\frac{2pr\sqrt{-6pq}}{5q}\chi_x\varphi_y f(\xi)g(\xi), \end{aligned} \quad (13)$$

where $p = \pm 1$, $\varepsilon = \pm 1$, q and r are arbitrary constants, f and g are expressed by (5).

(c) For $\delta = h^2 - s^2$ and $pq < 0$, the solitary wave solutions are

$$\begin{aligned} u_{31} &= \frac{\varepsilon\chi_{xx} - \chi_t}{\chi_x} + \frac{\sqrt{pq(s^2 - h^2 - r^2)}}{q}\chi_x f(\xi) \\ &\quad + \varepsilon p\chi_x g(\xi), \\ v_{31} &= -pr\chi_x\varphi_y f(\xi) - \frac{p(s^2 - h^2 - r^2)}{q}\chi_x\varphi_y f^2(\xi) \\ &\quad - \varepsilon\frac{p\sqrt{pq(s^2 - h^2 - r^2)}}{q}\chi_x\varphi_y f(\xi)g(\xi); \end{aligned} \quad (14)$$

$$\begin{aligned} u_{32} &= \frac{\varepsilon\chi_{xx} - \chi_t}{\chi_x} - \frac{\sqrt{pq(s^2 - h^2 - r^2)}}{q}\chi_x f(\xi) \\ &\quad + p\chi_x g(\xi), \\ v_{32} &= -pr\chi_x\varphi_y f(\xi) - \frac{p(s^2 - h^2 - r^2)}{q}\chi_x\varphi_y f^2(\xi) \\ &\quad + \varepsilon\frac{p\sqrt{pq(s^2 - h^2 - r^2)}}{q}\chi_x\varphi_y f(\xi)g(\xi), \end{aligned} \quad (15)$$

where $p = \pm 1$, $\varepsilon = \pm 1$, s, r and h are arbitrary constants, f and g are expressed by (6).

(d) For $\delta = -h^2 - s^2$ and $pq > 0$, the trigonometric function solutions are

$$\begin{aligned} u_{41} &= \frac{\chi_{xx} - \chi_t}{\chi_x} + \varepsilon\frac{\sqrt{pq(h^2 + s^2 - r^2)}}{q}\chi_x f(\xi) \\ &\quad + p\chi_x g(\xi), \\ v_{41} &= -pr\chi_x\varphi_y - \frac{p(h^2 + s^2 - r^2)}{q}\chi_x\varphi_y f^2(\xi) \\ &\quad - \frac{p\sqrt{pq(h^2 + s^2 - r^2)}}{q}\chi_x\varphi_y f(\xi)g(\xi); \end{aligned} \quad (16)$$

$$\begin{aligned} u_{42} &= -\frac{\chi_{xx} + \chi_t}{\chi_x} + \varepsilon\frac{\sqrt{pq(h^2 + s^2 - r^2)}}{q}\chi_x f(\xi) \\ &\quad - p\chi_x g(\xi), \\ v_{42} &= -pr\chi_x\varphi_y - \frac{p(h^2 + s^2 - r^2)}{q}\chi_x\varphi_y f^2(\xi) \\ &\quad + \varepsilon\frac{p\sqrt{pq(h^2 + s^2 - r^2)}}{q}\chi_x\varphi_y f(\xi)g(\xi), \end{aligned} \quad (17)$$

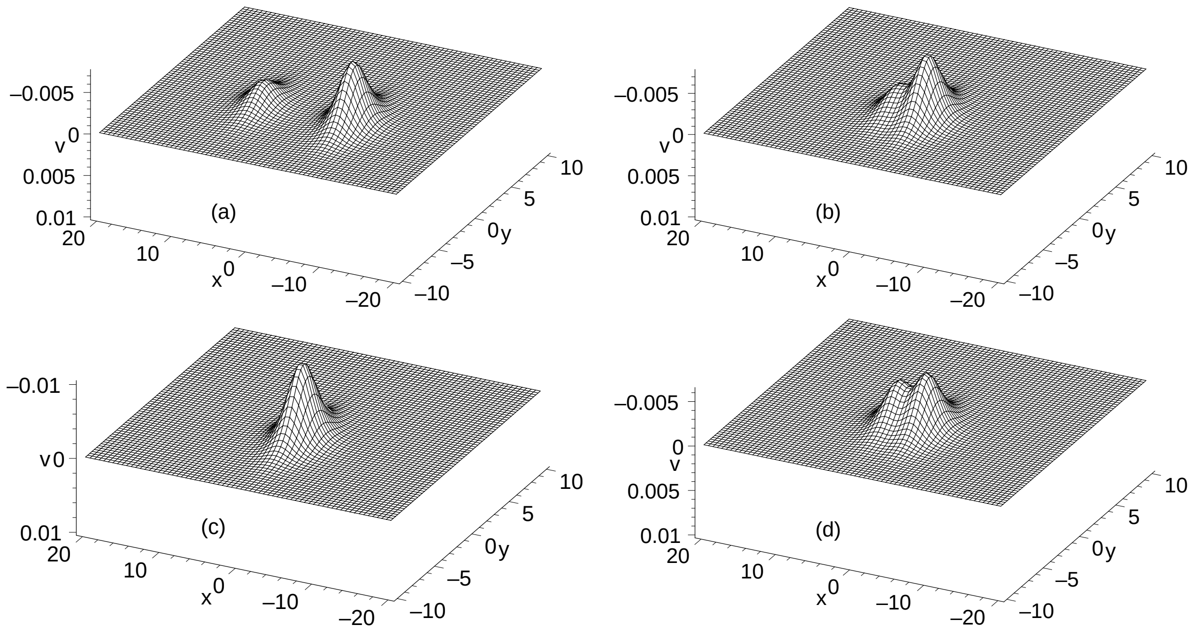
where $p = \pm 1, \varepsilon = \pm 1$, s, r and h are arbitrary constants, f and g are expressed by (7).

(e) For $q = 0$, the rational solutions are

$$\begin{aligned} u_{51} &= \frac{\varepsilon\chi_{xx} - \chi_t}{\chi_x} + \frac{p}{2}\sqrt{C_1^2 + 4C_2pr}\chi_x f(\xi) \\ &\quad + \varepsilon p\chi_x g(\xi), \\ v_{51} &= -pr\chi_x\varphi_y - \frac{p^2}{4}(C_1^2 + 4C_2pr)\chi_x\varphi_y f^2(\xi) \\ &\quad - \varepsilon\frac{p^2}{2}\sqrt{C_1^2 + 4C_2pr}\chi_x\varphi_y f(\xi)g(\xi); \end{aligned} \quad (18)$$

$$\begin{aligned} u_{52} &= \frac{\varepsilon\chi_{xx} - \chi_t}{\chi_x} - \frac{p\sqrt{C_1^2 + 4C_2pr}}{2}\chi_x f(\xi) \\ &\quad + \varepsilon p\chi_x g(\xi), \\ v_{52} &= -pr\chi_x\varphi_y - \frac{p^2}{4}(C_1^2 + 4C_2pr)\chi_x\varphi_y f^2(\xi) \\ &\quad + \varepsilon\frac{p^2}{2}\sqrt{C_1^2 + 4C_2pr}\chi_x\varphi_y f(\xi)g(\xi), \end{aligned} \quad (19)$$

where C_1, C_2, and r are arbitrary constants, $p = \pm 1$, $\varepsilon = \pm 1$, f and g are expressed by (8).



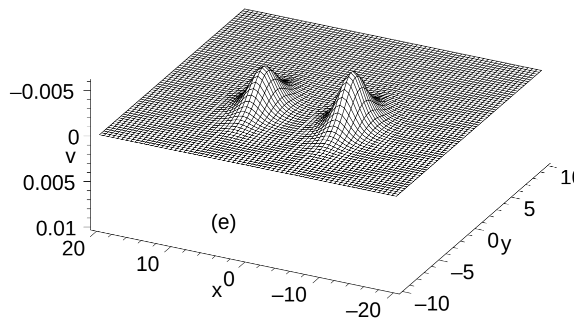


Fig. 1. Evolutional profiles of two travelling dromions for the solution v expressed by (20) with condition (22) at different times: (a) $t=-3$; (b) $t=-1$; (c) $t=0$; (d) $t=1$; (e) $t=3$.

Comparing the above derived exact solutions with the solutions obtained in [12], we can find that the solutions reported in [12] are special situations of the cases (c), (d), (e) as the parameters r, s, h, p, C_1, C_2 are chosen as some determinate values, while for the cases (a), (b), they are new solutions.

3. Some Special Localized Excitations in the DLW System

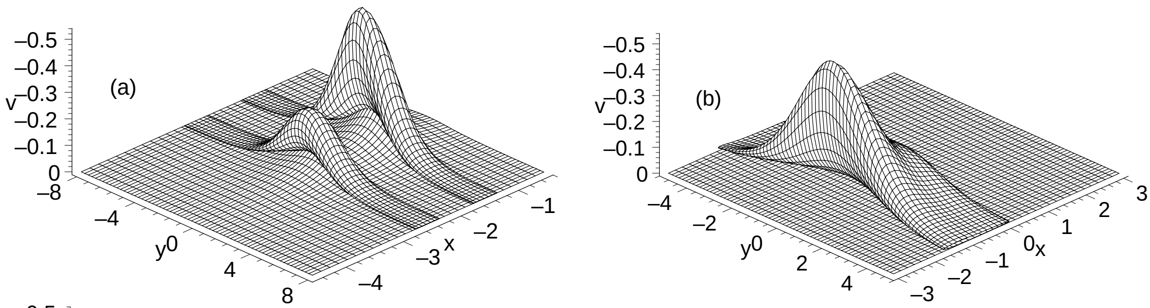
Since some arbitrariness of the functions $\chi(x,t)$ and $\varphi(y)$ is included in the above cases, the physical quantities u and v may possess rich structures. For example, when $\chi(x,t)=f(kx+ct)$, where k and c are arbitrary constants, and f is an arbitrary function of the indicated argument, one of the simplest travelling wave solutions can be easily obtained. In the following part we merely analyze some localized excitations of solution v_{51} (18) and rewrite it in a simple form (as $p=1$, $r=0$, $C_1=1$, $\varepsilon=1$ and $C_2=-1$), namely

$$v \equiv v_{51} = -2\frac{\chi_x\varphi_y}{(1+\chi+\varphi)^2}. \tag{20}$$

3.1. Single-Valued Localized Excitations

It is well known that in (2+1)-dimensional cases, one of the significant localized excitations is the dromion-type coherent solution, which is localized exponentially in all directions. If the choice of the functions χ and φ is considered to be

$$\begin{aligned}\chi &= a_0+\sum_{i=1}^{M} a_i\tanh(k_ix+x_{i0}t),\\ \varphi &= b_0+\sum_{i}^{N} b_i\tanh(K_iy+y_{i0}),\end{aligned} \tag{21}$$



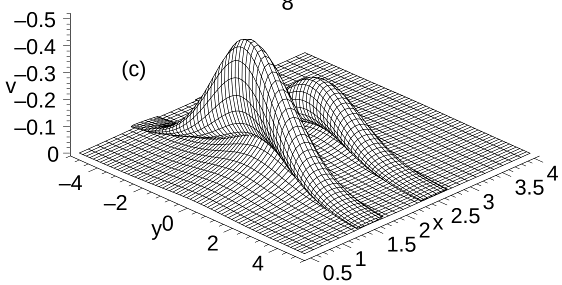


Fig. 2. Evolutional profiles of the interaction between two semifolded localized structures for v expressed by (20) with condition (24) at different times: (a) $t = -10$; (b) $t = -4$; (c) $t = 10$.

where K_i, k_i, x_{i0}, y_{i0}, a_0, b_0, a_i, b_i are all arbitrary constants, then the quantity v_{51} (20) denotes a $M \times N$ dromion solution. The $M \times N$ dromions are located at the $M \times N$ "lattice" and the $M \times N$ lattice points are just the cross points of $M+N$ "ghost" straight line solitons. For example, if we take

$$\chi = 1 + 0.7\tanh(0.5x+t) + 0.5\tanh(0.5x-t),\\ \varphi = 2 + 0.25\tanh(0.5y+1), \tag{22}$$

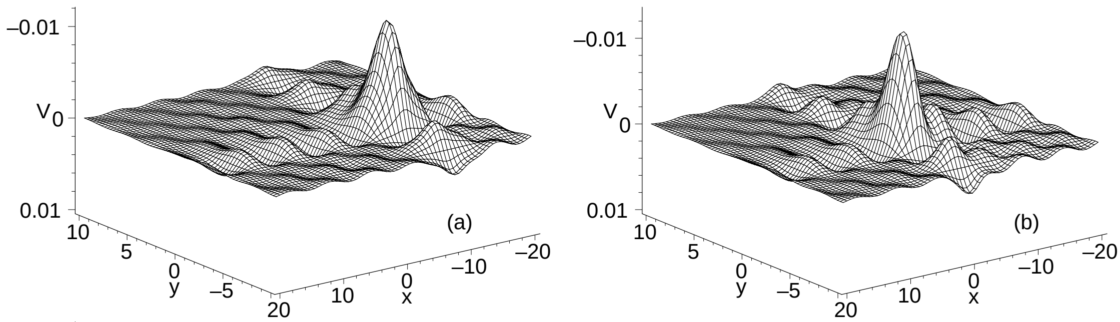
then we can obtain a travelling two-dromion excitation for the field v expressed by (20). The corresponding evolutional plots are presented in Figure 1.

3.2. Multiple-Valued Localized Excitations

In addition to the single-valued localized excitations, there are some multiple-valued localized excitations in the natural world. In various real situations, it is impossible to describe the natural world only by single-valued functions. For instance, there are many complicated phenomena such as folded proteins, folded brain, folded skin surfaces, and many other kinds of folded biological systems in the real world. To study these complicated folded natural phenomena is very difficult. Actually, at the present stage, it is even impossible to give a relatively complete view on these complicated folded natural phenomena. However in the current paper, we may pay our attention to some simple and stable multiple-valued solitary waves such as semifolded localized structures. For example, some semifolded localized structures such as ocean waves may fold in one direction, say x, and are localized in a usual single-valued way in another direction, say y. Here we would list these intricate localized coherent structures for the field v (20) which may exist in certain situations: when the function φ is t-independent and χ is chosen via the relations

$$\chi_x = \sum_{j=1}^{M} P_j(\zeta + c_j t), \quad x = \zeta + \sum_{j=1}^{M} Q_j(\zeta + c_j t),\\ \chi = \int^{\zeta} \chi_x x_\zeta \, \mathrm{d}\zeta, \tag{23}$$

where the c_j $(j = 1,2,\cdots,M)$ are arbitrary constants and P_j, Q_j are localized excitations with the properties $P_j\ (\pm\infty) = 0$, $Q_j\ (\pm\infty) = \text{const}$. From (23), one knows that ζ may be a multi-valued function in some suitable regions of x by choosing the function Q_j appropriately. Therefore, the function χ_x may be a multi-valued function of x in some intervals though it is a single-valued function of ζ in others. Moreover, in general terms, if the functions χ and φ are taken as multiple-localized excitations, that possess phase shifts of (1+1)-dimensional models, then the three-dimensional localized excitations involving representation (20) inherit a phase shift structure. For example,



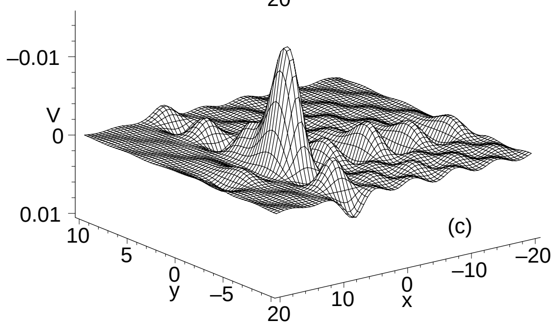


Fig. 3. Evolutional behavior of two solitary waves in a periodic wave background for the field v_1 expressed by (20) with condition (25) at different times: (a) $t=-4$; (b) $t=0$; (c) $t=4$.

when

$$\begin{aligned}&\chi_x=\mathrm{sech}^2(\zeta)+0.5\mathrm{sech}^2(\zeta-0.4t),\\&x=\zeta-1.5\tanh(\zeta)-1.5\tanh(\zeta-0.4t),\\&\varphi=\tanh(ky),\end{aligned}\tag{24}$$

then we obtain two semifolded localized excitations. Some interesting structures are depicted in Fig. 2 $(k=1)$. From Fig. 2, one finds that the two semifolded structures possess novel properties, which fold in the x-direction, and localize in a single-valued way in the y-direction. The characteristic width of the y-localization is determined by the coefficient k (k is a nonzero constant), namely, the width of the y-localization increases as k decreases. Moreover, one may find that the interaction between the two semifolded localized structures (also called foldons) is completely elastic, since the velocities, amplitudes, and shapes are completely preserved after their interaction.

3.3. Complex Wave Excitations: Solitary Wave in Periodic Waves

The above localized excitations are propagating on a constant ideal background. Actually, these cases do not exist in reality since some background waves are always encountered. In order to simulate some real waves such as ocean waves, we give a simple complex wave excitation for the DLW system: solitary wave in periodic wave background. Considering the arbitrariness of χ and φ, we take them as

$$\begin{aligned}&\chi(x,t)=2+0.2J(2,x)+0.75\tanh(0.5x-t),\\&\varphi(y)=2+0.2J(2,y)+0.75\tanh(0.5y+1),\end{aligned}\tag{25}$$

where J is a Bessel function of the indicated variables. Then we can reveal a complex wave for the field v in the DLW system. Figure 3 shows the corresponding profiles of the complex wave excitation presenting the propagation of a dromion moving along the x-axis in the positive direction in the determined periodic wave background. As shown in Fig. 3, during the process of propagation the amplitude of the dromion changes due to the superposition of the solitary wave and underlying periodic wave. If the background wave amplitude in Fig. 3 increases and/or the soliton amplitude increases, we find that the complex wave amplitude increases based on wave superposition theorem. However, its wave shape and wave velocity do not suffer any change, which is very close to many actual physical processes in the natural world.

4. Summary and Conclusion

In summary, with the aid of the improved mapping method and the linear variable separation method, analytical investigation of the (2+1)-dimensional DLW system shows the existence of interacting coherent excitations such as single-valued localized excitations and multiple-valued localized excitations revealed by prescribing appropriate functions in terms of the derived variable separation solutions. Meanwhile, we have obtained some complex wave excitations in the (2+1)-dimensional DLW system, which describe some solitary waves moving on a periodic wave background. Some interesting evolutional properties similar to the elastic interaction for the multiple-valued localized excitations and complex wave excitations are briefly discussed. We hope these single-valued localized excitations, multiple-valued localized excitations, and complex wave excitations would be helpful to some practical applications in reality.

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